**END 2962 NUMERICAL ANALYSIS**

**LEC #3**

**solutıon of equations wıth one varıable**

# Bracketing Methods

This lecture on roots of equations deals with methods that exploit the fact that a function typically changes sign in the vicinity of a root. These techniques are called bracketing methods because two initial guesses for the root are required. As the name implies, these guesses must “bracket,” or be on either side of, the root. The particular methods described herein employ different strategies to systematically reduce the width of the bracket and, hence, home in on the correct answer.

# Graphical Methods

A simple method for obtaining an estimate of the root of the equation is to make a plot of the function and observe where it crosses the axis. This point provides a rough approximation of the root.

Graphical techniques are of limited practical value because they are not precise. However, graphical methods can be utilized to obtain rough estimates of roots. These estimates can be employed as starting guesses for numerical methods. Aside from providing rough estimates of the root, graphical interpretations are important tools for understanding the properties of the functions and anticipating the pitfalls of the numerical methods. Fig. 1 shows a number of ways in which roots can occur (or be absent) in an interval prescribed by a lower bound xl and an upper bound xu.

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Figure 1: Illustration of a number of general ways that a root may occur in an interval prescribed by a lower bound and an upper bound .

# The Bisection Method

When applying the graphical technique, you have observed that f(x) changed sign on opposite sides of the root. In general, if f (x) is real and continuous in the interval from to and and have opposite signs (Eq.1), and then there is at least one real root between and .

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Incremental search methods capitalize on this observation by locating an interval where the function changes sign. Then, the location of the sign change identifies the root precisely by dividing the interval into a number of subintervals. Each of these subintervals is searched to locate the sign change. The process is repeated and the root estimate refined by dividing the subintervals into finer increments. The **bisection method**, one type of incremental search method in which the interval is always divided in half.



Suppose is a continuous function defined on the interval [a, b], with and of opposite sign. The **Intermediate Value Theorem** implies that a number ***p*** exists in (a, b) with 0. Although the procedure will work when there is more than one root in the interval (a, b), we assume for simplicity that the root in this interval is unique.

*Step 1:* Check to ensure that the function changes sign over the interval. *Step 2:* Set and and let be the midpoint of [a, b]:

*Step 3:* Make the following evaluations to determine in which subinterval to root lies:

1. If , then , and we are done.
2. If , then ) has the same sign as either or .
   * If ) and have the same sign, Set and .
   * If ) and have opposite signs, Set and .
   * Then reapply the process to the interval

**Example 1:** Use bisection method to find the roots of in the interval

**Example 2:** Show that has a root in[1,2] ,and use the Bisection method to determine an approximation to the root that is accurate to at least within 10−4.

**Homework:** Use graphical approach and bisection method to determine the drag coefficient c in falling parachutist example when 68.1 kg parachutist has the velocity of 40 m/s after free falling for time 10s. ()

## **Termination Criteria & Error Estimates**

We must now develop an objective criterion for deciding when to terminate the method. An initial suggestion might be to end the calculation when true error falls below some pre-specified level. We might decide that we should terminate when the error drops below. Therefore; we require an error estimate that is not contingent on foreknowledge of the root. An approximate percent relative error can be calculated as;

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where is the root for the present iteration and is the root from the previous iteration. The absolute value is used because we are concerned with the magnitude of rather than with its sign. When becomes less than a pre-specified stopping criterion , the computation is terminated.

The Bisection method, though conceptually clear, has significant drawbacks. It is relatively slow to converge (that is, N may become quite large before is sufficiently small), and a good intermediate approximation might be inadvertently discarded. However, the method has the important property that it always converges to a solution, and for that reason it is often used as a starter for the more efficient methods.

**Theorem 1:** Suppose that and . The Bisection method generates a sequence approximating a zero of with

or with notation p:

**Proof:** For each we have

Since for all

Then the sequence converges to with the rate of convergence .

It is important to realize that Theorem 1 gives only a bound for approximation error and that this bound might be quite conservative. For example, this bound applied to the problem in Example 1 ensures only that but the actual error is much smaller

**Example 3:** Determine the number of iterations necessary to solve has a root in[1,2] with the accuracy 10−3.

## **Bisection Algorithm**

Here pseudo codes are represented to implement Bisection Algorithm. Most general sense, a univariate function is merely an entity that returns a single value in return for a single value you send to it. Perceived in this sense, functions are not always simple formulas like the one-line equations. A function might consist of many lines of code that could take a significant amount of execution time to evaluate**.**

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# THE FALSE-POSITION METHOD

A shortcoming of the bisection method is that, in dividing the interval from to xu into equal halves, *no account is taken of the magnitudes of f (xl) and f (xu)*. For example, if f (xl) is much closer to zero than f (xu), it is likely that the root is closer to xl than to xu.

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Figure 2: A graphical depiction of the method of false position.

Similar triangles used to derive the formula for the method are shaded.

An alternative method that exploits this graphical insight is to join f (xl) and f (xu) by a straight line. The intersection of this line with the x axis represents an improved estimate of the root. The fact that the replacement of the curve by a straight line gives a “false position” of the root is the origin of the name, method of false position, or in Latin, regula falsi. It is also called the linear interpolation method.

Using similar triangles (Fig.2), the intersection of the straight line with the x axis can be estimated as

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which can be solved for

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This is the *false-position formula*. The value of xr computed with Eq.4 then replaces whichever of the two initial guesses, xl or xu, yields a function value with the same sign as f (xr). In this way, the values of xl and xu always bracket the true root. The process is repeated until the root is estimated adequately. The algorithm is identical to the bisection algorithm with the exception that Eq.4 is used for step 2. In addition, the *same stopping criterion [Eq.2]* is used to terminate the computation.

**Example 3:** Use regula-falsi method to find the roots of in the interval

**Example 4:** Use false position method to determine the drag coefficient c in falling parachutist example when 68.1 kg parachutist has the velocity of 40 m/s after free falling for time 10s. ()

**Example 5:** Use the method of False Position to find a solution to in the interval

## **Pitfalls of False Position Method**

Although the false-position method would seem to always be the bracketing method of preference, there are cases where it performs poorly. In fact, as in the following example, there are certain cases where bisection yields superior results.

**Example 5:** Use bisection and false position to locate the root of f(x) = x10 – 1 between x=0 and 1.3.



Thus, after five iterations, the true error is reduced to less than 2 percent. For false position, a very different outcome is obtained:



The example also illustrates a major weakness of the false-position method: its one-sidedness. That is, as iterations are proceeding, one of the bracketing points will tend to stay fixed. This can lead to poor convergence for functions with significant curvature.

One way to mitigate the “one-sided” nature of false position is to have the algorithm detect when one of the bounds is stuck. If this occurs, the function value at the stagnant bound can be divided in half. This is called the modified false-position method.

